Math 254-1 Exam 5 Solutions

1. Carefully state the definition of "basis". Give two examples from $\mathbb{R}^2$.

   A basis is a set of vectors that is both independent and spanning. Equivalently, a basis is a maximal set of independent vectors. Equivalently, a basis is a minimal set of spanning vectors. Many examples are possible, such as $\{(1,0),(0,1)\}$ or $\{(1,1),(2,3)\}$. All must contain exactly two, linearly independent, vectors.

Problems 2 and 3 both concern the matrix $A = \begin{pmatrix} 1 & -2 & 1 & -1 & 1 \\ 2 & -4 & 3 & 0 & -1 \\ 1 & -2 & 0 & -3 & 4 \\ -1 & 2 & 1 & 5 & -7 \\ -2 & 4 & -4 & -2 & 4 \end{pmatrix}$.

2. Set $S = Rowspace(A)$. Find a basis for S, and determine its dimension.

   $\begin{pmatrix} 1 & -2 & 1 & -1 & 1 \\ 2 & -4 & 3 & 0 & -1 \\ 1 & -2 & 0 & -3 & 4 \\ -1 & 2 & 1 & 5 & -7 \\ -2 & 4 & -4 & -2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & -1 & -2 & 3 \\ 0 & 0 & 2 & 4 & -6 \\ 0 & 0 & -2 & -4 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, in row echelon form. There are two pivots, hence S is two dimensional. Many bases are possible; the natural one is the nonzero rows of the echelon matrix: $\{(1,-2,1,-1,1),(0,0,1,2,-3)\}$. However any two independent elements of the rowspace would also work, such as two rows of A itself: $\{(2,-4,3,0,-1),(-2,4,-4,-2,4)\}$.

3. Set $T = Columnspace(A)$. Find a basis for T, and determine its dimension.

   The rowspace and columnspace have the same dimension, hence T is two dimensional. The pivots of the row echelon form of A are in the first and third columns, hence the first and third columns of A form a basis for the columnspace: $\{(1,2,1,-1,-2)^T,(1,3,0,1,-4)^T\}$. This is not the only basis; any two independent elements of the columnspace would also work, such as $\{(-1,0,-3,5,-2)^T,(1,-1,4,-7,4)^T\}$. The first two columns of A will NOT work, since they are linearly dependent.

Problems 4 and 5 both concern the vector spaces $A = Span\big((1,2,3),(-1,1,2)\big)$ and $B = Span\big((5,1,0),(9,0,-3)\big)$. Both are subspaces of $\mathbb{R}^3$.

4. Find a basis for $A+B$, and determine its dimension.

   We begin by putting the generating vectors in a matrix, then putting this matrix into echelon form: $\begin{pmatrix} 1 & -1 & 5 & 9 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 5 & 9 \\ 0 & 3 & -9 & -18 \\ 0 & 5 & -15 & -30 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 5 & 9 \\ 0 & 3 & -9 & -18 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. This has two pivots, hence $A+B$ is two dimensional. The pivots are in the first two columns, hence a basis is $\{(1,2,3),(-1,1,2)\}$ (other bases are possible). Note: this solution has the matrix as 3×4, putting the vectors into columns. It is equally correct to put the vectors into rows, giving a 4×3 matrix.

5. Find a basis for $A\cap B$, and determine its dimension.

   $dim(A+B) + dim(A\cap B) = dim(A) + dim(B)$. We have $dim(A) = dim(B) = dim(A+B) = 2$, hence we can solve and determine $dim(A\cap B) = 2$. Hence $A\cap B = A = B = A+B$, so a basis for $A\cap B$ is any basis for A (or B), such as $\{(1,2,3),(-1,1,2)\}$.